MATH20132 Calculus of Several Variables. 2020-21

Problems 8 Tangent Spaces & Planes

Tangent Planes to level sets.

1. For each of the following level sets find the tangent plane to the surface at the given point $\mathbf{p}$ and give your answer *as a level set.*

i. $(x, y, z)^T \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 14$ with $\mathbf{p} = (2, 1, -3)^T$,

ii. $(x, y, z)^T \in \mathbb{R}^3$:

$$\begin{aligned} x^2 + 3y^2 + 2z^2 &= 9, \\ xyz &= -2, \end{aligned}$$

with $\mathbf{p} = (2, -1, 1)^T$,

iii. $(x, y, u, v) \in \mathbb{R}^4$:

$$\begin{aligned} x^3 - 3yu + u^2 + 2xv &= 12 \\ xv^2 + 2y^2 - 3u^2 - 3yv &= -3. \end{aligned}$$

with $\mathbf{p} = (1, 2, -1, 2)^T$,

2. Return to your answers of Question 1 and write them as graphs instead of level sets. Then give a basis for the Tangent Space.

Tangent Spaces for Image sets.

3. In each case, find parametric equations for the Tangent Plane passing through the point $\mathbf{F}(\mathbf{q})$ on the parametric surfaces given by the following functions.

i. $\mathbf{F}\big((x,y)^T\big) = (x^2+y^2,\, xy,\, 2x-3y)^T, \qquad \text{at } \mathbf{q}=(1,2)^T,$

ii. $\mathbf{F}\big((x,y)^T\big) = (xy^2,\, x^2+y,\, x^3-y^2,\, y^2)^T, \qquad \text{at } \mathbf{q}=(-1,2)^T,$

iii. $\mathbf{F}(t) = (\cos t,\, \sin t,\, t)^T$ at $q = 3\pi$.

4. Return to Question 7 on Sheet 6. You were asked to show, by using the Implicit Function Theorem, that the following equations

$$\begin{aligned} x^2+y^2+2uv &= 4 \\ x^3+y^3+u^3-v^3 &= 0, \end{aligned} \tag{1}$$

determine u and v as functions of x and y for $(x,y)^T$ in an open subset of $\mathbb{R}^2$ containing the point $\mathbf{q}=(-1,1)^T \in \mathbb{R}^2$. The implicit function theorem is an existence result, it does not say what u and v are as functions of x and y. Nonetheless it is possible to find their partial derivatives and you were asked to do this. The answer was

$$\frac{\partial u}{\partial x}(\mathbf{q})=0,\ \frac{\partial v}{\partial x}(\mathbf{q})=1,\ \frac{\partial u}{\partial y}(\mathbf{q})=-1 \quad \text{and} \quad \frac{\partial v}{\partial y}(\mathbf{q})=0.$$

Use these partial derivatives to find a basis for the tangent space at $\mathbf{p} = (-1,1,1,1)^T$.

5. Let $S(\mathbf{u}) = (\cos u \sin v,\, \sin u \sin v,\, \cos v)^T$, where $\mathbf{u}=(u,v)^T$, with $0 \le v \le \pi$, $0 \le u \le 2\pi$. This is the surface of the unit ball in $\mathbb{R}^3$ in standard spherical coordinates.

i. Show that the tangent space of S at $\mathbf{q}=(\pi,\, \pi/2)^T$ is $T_{\mathbf{p}}S = \text{Span}(\mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{p}=S(\mathbf{q})$.

ii. Determine also the tangent space at $\mathbf{q}=(0,\, \pi/4)^T$.

iii. a. Let $\mathbf{w}=(1,2,-1)^T/\sqrt{6}$. Show that $\mathbf{w} \in T_{\mathbf{p}}S$ where $\mathbf{p}=S\big((0,\, \pi/4)^T\big)$.

b. (Tricky) The definition of $T_{\mathbf{p}}S$ is that $\mathbf{w} \in T_{\mathbf{p}}S$ only if there exists a curve $a : I \to S$ such that $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{w}$. Find a α in this case.

Hint In the notes we prove that $T_{\mathbf{p}}(S) = \{J\mathbf{F}(\mathbf{q})\,\mathbf{x}\}$ when $S = \operatorname{Im}\mathbf{F}$. Look at that proof which constructs a curve within the surface.

Additional Questions

6 Assume $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a C^1-function on U. Assume that at $\mathbf{a} \in U$ the Jacobian matrix $J\mathbf{f}(\mathbf{a})$ is of full-rank. Prove that there exists an open set $A : \mathbf{a} \in A \subseteq U$ such that $J\mathbf{f}(\mathbf{x})$ is of full-rank for all $\mathbf{x} \in A$.

7 Let $C \subseteq \mathbb{R}^3$ be the level set

$$\begin{aligned} x^2z^3 - x^3z^2 &= 0, \\ x^2y + xy^3 &= 2. \end{aligned}$$

Show that in some neighbourhood of $\mathbf{p} = (1,1,1)^T$, C is a curve which can be parametrized by $\mathbf{g}(x) = (x, g_1(x), g_2(x))$ for differentiable functions g_1 and g_2.

Find a parametrization of the Tangent Line to C at $\mathbf{p}$.

8 Find the Tangent Plane to the surface

$$\begin{aligned} x^3 - y^3 + xv + uv &= 0, \\ xu^2 + yv^2 &= 0. \end{aligned}$$

where $(x,y,u,v)^T \in \mathbb{R}^4$, at $\mathbf{p} = (-1,1,-1,-1)^T$. Give your answer as a level set, and also as a graph. Find a basis for the Tangent Space to the surface at $\mathbf{p}$.

9 Find parametric equations for the tangent plane passing through the given point $\mathbf{F}(\mathbf{q})$ on the parametric surfaces given by

i. $\mathbf{F}\big((x,y)^T\big) = (x^2+y^2,\ xy,\ 2x-3y)^T$ at $\mathbf{q} = (1,1)^T$.

ii. $\mathbf{F}\big((s,t)^T\big) = (t\cos s,\ t\sin s,\ t)^T, \qquad \mathbf{q} = (\pi/2, 2)^T,$

iii. $\mathbf{F}\big((s,t)^T\big) = (t^2\cos s,\ t^2,\ t^2\sin s), \qquad \mathbf{q} = (0,1)^T,$

10 Find parametric equation for the Tangent Plane passing through the point $\mathbf{F}(\mathbf{q})$ on the parametric surface given by $\mathbf{F}(\mathbf{x}) = (yz, xz, xy, xyz)^T$, for $\mathbf{x} = (x,y,z)^T$ at $\mathbf{q} = (1,-1,2)^T$.

11. Find the tangent planes at the points $\mathbf{p}_1 = (1/\sqrt{2}, 1/4, 1/4)$ and $\mathbf{p}_2 = (\sqrt{3}/2, 0, 1/4)$ on the ellipsoid $x^2 + 4y^2 + 4z^2 = 1$.

Find the line of intersection of these two planes.

12. i. Find the Tangent Plane to the surface $z = xe^y$ at the point $\mathbf{p} = (1, 0, 1)^T$ on the surface.

ii. The surfaces $x^2 + y^2 - z^2 = 1$ and $x + y + z = 5$ intersect in a curve Γ. Find the equation in parametric form of the tangent line to Γ at the point $(1, 2, 2)^T$.

13. i. Consider the surface $S = \{(x, y, z)^T \in \mathbb{R}^3 : xy = z\}$. Let $\mathbf{p} = (A, B, C)^T$ be a generic point of S. Find the Tangent Plane at $\mathbf{p}$.

ii. Show that the intersection of the Tangent Plane with S consists of two straight lines.